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Transverse vertices in electrodynamics and the gauge technique

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Abstract. The inclusion of transverse vertex corrections in the gauge technique, needed for the restoration of gauge covariance, results in a self-consistent equation for the source propagator spectral function which agrees with perturbation theory when expanded to order e^4. We have managed to solve the equation in the infrared and ultraviolet limits, for scalar and spinor sources, in an arbitrary covariant gauge. The gauge covariance at asymptopia is thus established.

1. Introduction

As a non-perturbative approach to gauge theories the gauge technique (Delbourgo and Salam 1964, Strathdee 1964) transcends the original Baker–Johnson–Willey (1964) scheme. The notable successes of the technique (Delbourgo 1979) have largely stemmed from first gauge approximation (Delbourgo and West 1977a, b) whereby transverse corrections to the Green functions are neglected. While this may be fine at low- and high-energy limits, it is not at intermediate energies (Slim 1981, Delbourgo *et al* 1981) and it becomes desirable to improve the results by going to the next gauge approximation taking account of the transverse amplitudes in the vertex (and thus incorporating the charge and magnetic form factors etc). Apart from restoring gauge covariance at all momenta, the need for transverse vertices is most pressing in two- and three-dimensional theories (Gardner 1981, Roo and Stam 1984) where the vector particle acquires a mass (Deser *et al* 1982). In fact, in the two-dimensional case the exact form of the transverse vertex can be deduced from the axial gauge identities (Delbourgo and Thompson 1982).

For four-dimensional electrodynamics there have been two notable attempts at incorporating transverse vertices into the technique. King (1983), by looking at leading logarithmic terms in perturbation theory, introduced an approximate transverse vertex in QED which was asymptotically gauge covariant and which exactly renormalised the Dyson–Schwinger equation. However, the attempt was deficient in respect of the low-energy properties: the transverse vertex did not agree even with e^2 perturbation theory and had incorrect gauge dependence. Parker (1984) largely rectified these faults in the way he introduced transverse corrections, though only for scalar electrodynamics and only in the Fermi gauge. The purpose of this paper is to extend Parker's work to the spinor case and to allow for all possible gauge parameters. In the end we shall obtain a self-consistent equation for the propagator incorporating the transverse vertex which, when expanded to order e^4 agrees *exactly* with perturbation theory.

The outline of the gauge technique in the next gauge approximation appears in § 2 and is applied in § 3, where we derive the self-consistent equation of the spinor spectral function. This equation is very complicated but becomes amenable in the infrared and ultraviolet limits, yielding the leading behaviours:

$$S(p) \sim (\gamma p - m)^{-1+(a-3)e^2/8\pi^2}$$

$$S(p) \sim \left(\frac{\gamma p}{(p^2)^{1-ae^2/16\pi^2}} + \frac{m}{(p^2)^{1+3e^2/16\pi^2}}\right)\left(1 - \frac{3e^2}{16\pi^2}\ln\frac{p^2}{m^2}\right).$$

Finally, we extend Parker's scalar electrodynamics work to arbitrary gauges in § 4. The purpose of appendix 1 is to substantiate the vanishing of the transverse Green function in the soft-photon limit (i.e. proving the absence of a soft-photon singularity even when electrons are off-shell); while appendix 2 contains a few details about the determination of the discontinuity in the fermion self-energy to order e^4, the kernel of our equation for the spectral function.

2. The technique to order e^4

There are three major steps in applying the technique to gauge theories:

(a) Setting up the Dyson–Schwinger (DS) equations for the Green function of interest for the gauge model in question.

(b) Solving the gauge identities (up to transverse corrections be it understood) that involve the Green function; in this connection spectral representations can often facilitate solution.

(c) Truncating the DS equation to a particular order in e so as to obtain a self-consistent equation for the amplitude in terms of which the (longitudinal) higher amplitudes are expressed.

Until recently, researchers had contented themselves with studying the two-point function and, in essence, were applying the gauge technique in its most primitive form. Thus they were led to a self-consistent, but non-perturbative, solution of the propagator which was only exact to order e^2 and which sacrificed gauge covariance (Slim 1981, Delbourgo *et al* 1981) except at the asymptopia.

In this paper we wish to include some degree of transversality into the amplitudes by extending Parker's (1984) recent work on scalar electrodynamics to the more realistic spinor case, QED. We shall ascend to the next level of the technique by studying the three-point vertex (the two-point function follows from it) and constructing an ansatz for it which is exact to order e^3 and which provides a non-perturbative solution for the fermion propagator that is exact† to order e^4. These improvements mean that we are incorporating magnetic effects in the technique for the first time and, by the very inclusion of the transverse vertices, can look forward to an amelioration of the gauge covariance properties at intermediate momentum values.

This is how the three basic tools are wielded in practice:

(a) Including sources, the QED action is

$$S = \int d^4x[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\gamma(\mathrm{i}\delta - eA) - m)\psi - (\delta A)^2/2a - \bar{j}\psi - \bar{\psi}j - J^\mu A_\mu] \tag{1}$$

in a class of covarient gauges parametrised by (a). (All fields and constants in (1) are

† Apart from internal vacuum polarisation corrections which do not affect the issue of gauge covariance and can be separately accounted for.

as yet unrenormalised.) The generating functional of connected Green functions

$$W(j,\bar{j},J) = -\mathrm{i}\ln\int(\mathrm{d}A\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi})\exp(\mathrm{i}S)$$

satisfies the basic fermionic (DS) equation

$$\frac{\delta W}{\delta j(p)}(\gamma p - m_0) = e\int \bar{\mathrm{d}}^4k\left(\mathrm{i}\frac{\delta^2 W}{\delta J^\mu(k)\delta j(p-k)} - \frac{\delta W}{\delta J^\mu(k)}\frac{\delta W}{\delta j(p-k)}\right)\gamma_\mu - \bar{j}(p) \tag{2}$$

when expressed in momentum space. Additional j, J functional derivatives provide the DS equations of all fermionic amplitudes.

It is convenient to define photon amputated amplitudes $G(\equiv S\Gamma S)$ by

$$\begin{aligned}&\left.\frac{\delta^{n+2}W}{\delta\bar{j}(p')\delta j(p)\delta J_{\mu_1}(-k_1)\ldots\delta J_{\mu_n}(-k_n)}\right|_{j=J=0}\\ &\qquad=\bar{\delta}^4(p'-p-k_1-\ldots k_n)\\ &\qquad\quad\times(-e)^n D^{\mu_1\nu_1}(k_1)\ldots D^{\mu_n\nu_n}(k_n)G_{\nu_1\ldots\nu_n}(p',p;k_1\ldots k_n)\end{aligned} \tag{3}$$

where p' is outgoing, p and k_i are incoming momenta. Operating on (2) with $\delta^{n+1}/\delta\bar{j}(\delta J)^n$ with increasing n and, bearing in mind that

$$\delta^{n+1}W/(\delta J)^n\delta j = \delta^{n+1}W/(\delta J)^n\delta j = \delta W/\delta J = 0$$

for vanishing sources, one arrives at the successive DS equations:

$$S(p)(\gamma p - m_0) = Z^{-1} + \mathrm{i}e^2\int\bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)G_\kappa(p,p-k;k)\gamma_\lambda \tag{4}$$

$$\begin{aligned}&G_\mu(p',p;p'-p)(\gamma p - m_0)\\ &\qquad= S(p')\gamma_\mu - \mathrm{i}e^2\int\bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)G_{\mu\kappa}(p',p-k;p'-p,k)\gamma_\lambda\end{aligned} \tag{5}$$

$$\begin{aligned}&G_{\mu\nu}(p',p;q,p'-p-q)(\gamma p - m_0)\\ &\qquad= -G_\mu(p',p'-q;q)\gamma_\nu - G_\nu(p',p+q;p'-p-q)\gamma_\mu\\ &\qquad\quad -\mathrm{i}e^2\int\bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)G_{\mu\nu\kappa}(p',p-k;q,p'-p-q,k)\gamma_\lambda\end{aligned} \tag{6}$$

which are now renormalised. One may, of course, derive similar equations from the adjoint of (2) in which the differential operator $(\gamma p' - m_0)$ acts on the left. For instance:

$$\begin{aligned}&(\gamma p' - m_0)G_\mu(p',p;p'-p)\\ &\qquad= \gamma_\mu S(p) - \mathrm{i}e^2\int\bar{\mathrm{d}}^4k\, D^{\lambda\kappa}(k)\gamma_\lambda G_{\mu\kappa}(p'+k,p;p'-p,k).\end{aligned} \tag{5a}$$

Before moving to the next phase, it is worth observing that the gauge identities (arising by contractions with k) relate one equation to the next; in effect the lower point DS equations are subsumed in the higher point equations. For example, contracting (5) with k^μ yields (4).

(b) The next step is to exploit the gauge identities so as to gain some information about the longitudinal parts of Green functions on the RHS of (4), (5) and (6). The first few identities read:

$$(p'-p)^\lambda G_\lambda(p',p;p'-p)=S(p)-S(p') \tag{7}$$

$$k^\lambda G_{\lambda\mu}(p',p;k,k')=G_\nu(p',p+k;k')-G_\nu(p'-k,p;k') \tag{8}$$

$$k^\lambda G_{\lambda\mu\nu}(p',p;k,k',k'')=G_{\mu\nu}(p',p+k;k',k'')-G_{\mu\nu}(p'-k,p;k',k'') \tag{9}$$

and in the limit as the photon momentum vanishes they lead to the differential forms of the identities:

$$G_\lambda(p,p;0)=-\partial S(p)/\partial p_\lambda \qquad \text{etc.}$$

From these relations one may abstract the longitudinal parts of the n-point G (those that survive contraction with k) in terms of the full $(n-1)$-point G. The transverse parts G^{T} remain undetermined of course, but if we insist that the longitudinal G^{L} are non-singular and obey the differential identities then we can safely disregard the G^{T} in the infrared domain†. Actually one can reconstruct a good measure of the amplitudes in terms of the propagator S by using the exact spectral representation

$$S(p)=\int \rho(W)\,\mathrm{d}W/(\gamma p-W) \tag{10}$$

and making the ansätze

$$G^{\mathrm{L}}_{\mu_1\ldots\mu_n}(p',p;k_1\ldots k_n)=\int \rho(W)\,\mathrm{d}W\,G^0_{\mu_1\ldots\mu_n}(p',p;k_1\ldots k_n|W) \tag{11}$$

where $G^0(\,|W)$ is the Born amplitude for a fermion of mass W. Not only are all gauge identities automatically respected and the infrared properties exact, but the lowest-order perturbation results are naturally incorporated. Put another way, the longitudinal amplitudes (11), including coupling factors, are exact to order e^n and do not affect the analytic properties anticipated from Feynman diagrammatics. We could sharpen the G ansätze if general spectral representations were available for all momenta off-shell, allowing solution of the identities at higher levels. Unfortunately, even for the three-point function, a totally general and usable representation does not exist; however, we are fortunately able to overcome this problem below.

(c) A *closed* form equation for the amplitudes is attained if one *truncates* the DS equations at some level. Heretofore almost all research has been directed at the primitive equation (4) in the first gauge approximation $G_\mu\to G^{\mathrm{L}}_\mu[S]$, $D_{\mu\nu}(k)\to D^{\mathrm{bare}}_{\mu\nu}(k)$. Although the resulting non-perturbative solutions are endowed with many attractive features (correct analyticity, asymptotics, exact infrared behaviour), they are also deficient in many respects, primarily through the neglect of G^{T} in (4). This is what we shall try to remedy here.

The idea is to move up to the next level equations (5) and (8) where the full three-point vertex is determined by the four-point amplitude. In principle (8) gives the longitudinal $G_{\mu\nu}$ in terms of the complete G_μ which can then be substituted into (5)—the second gauge approximation—so as to yield a self-consistent vertex equation‡

† And quite often in the ultraviolet too as we shall see; at least the G^{T} do not spoil the asymptotic gauge covariance of the amplitudes.

‡ These steps can probably be carried out, complicated though they certainly are. The subsequent task of solving for G_μ is even more difficult, if not intractable. Since we already know that $\mathrm{O}(e^6)$ corrections to S must come in through $G^{\mathrm{T}}_{\mu\nu}$, which modify the resulting G_μ, it makes as much sense to follow the simpler approach advocated below with no sacrifice in the degree of accuracy.

from which S can be extracted. Since transverse corrections of order e^4 are neglected by dropping $G^{\mathrm{T}}_{\mu\nu}$ it follows that the resulting Γ can only be precisely correct to order e^4. In practice then, to the same order of accuracy, it is sufficient to adopt the ansatz (11)

$$G^{\mathrm{L}}_{\mu\nu}=\int \mathrm{d}W\rho(W)(\gamma p'-W)^{-1}\{\gamma_\nu[\gamma(p+k)-W]^{-1}\gamma_\mu + \gamma_\mu[\gamma(p+k')-W]^{-1}\gamma_\nu\}(\gamma p-W)^{-1} \tag{12}$$

before substituting in (8). The resulting non-perturbative ρ (or propagator S) will then be correct to fourth order; and gauge covariance will be restored to the same order with (one may hope) a corresponding improvement in the momentum at non-asymptotic values.

This is the general strategy, which is executed in the following section.

3. The self-consistent equation for ρ

To arrive at the improved spectral equation we must substitute for $G_{\mu\nu}$ in (4) using (5) and (8). This is practicable (and correct to order e^4) when we use the longitudinal ansatz $G^{\mathrm{L}}_{\mu\nu}$, equation (12), on the right of (5). Take the difference of (5) and its adjoint

$$\begin{aligned}\gamma p' G_\mu - G_\mu \gamma p = \gamma_\mu S(p) - S(p')\gamma_\mu &\\ -\mathrm{i}e^2\int \bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)[\gamma_\lambda G^{\mathrm{L}}_{\mu\kappa}(p'+k,p;p'-p,k)&\\ - G^{\mathrm{L}}_{\mu\kappa}(p',p-k;p'-p,k)\gamma_\lambda]&\\ +\mathrm{O}(e^4)\text{ terms from } G^{\mathrm{T}}.&\end{aligned}$$

Next, multiply the LHS by $\gamma p'$, the RHS by γp, take the difference and use the spectral representations for S and G^{L}. This yields the *exact* expression

$$G_\mu(p',p)=\int \mathrm{d}W\frac{\rho(W)}{\gamma p'-W}\left(\gamma_\mu+\Lambda^2_\mu(p',p|W)+\frac{(\gamma p'\gamma_\mu+\gamma_\mu\gamma p)}{p'^2-p^2}\Sigma^2(p|W) - \Sigma^2(p'|W)\frac{(\gamma p'\gamma_\mu+\gamma_\mu\gamma p)}{p'^2-p^2}\right)\frac{1}{\gamma p-W}+\mathrm{O}(e^4) \tag{13}$$

where

$$\Lambda^2_\mu(p',p|W)=\mathrm{i}e^2\int \bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)\gamma_\kappa\frac{1}{\gamma(p'-k)-W}\gamma_\mu\frac{1}{\gamma(p-k)-W}\gamma_\lambda \tag{14}$$

$$\Sigma^2(p|W)=-\mathrm{i}e^2\int \gamma_\kappa\frac{1}{\gamma(p-k)-W}\gamma_\lambda D^{\kappa\lambda}(k)\,\bar{\mathrm{d}}^4k \tag{14a}$$

are the order e^2 perturbative results for the vertex and self-energy of a mass W electron. The second gauge approximation consists in dropping the finite $\mathrm{O}(e^4)$ correction on the RHS of (13). It is important to realise that, even so, (13) does include transversal corrections to G_μ; indeed if we identify

$$G^{\mathrm{L}}_\mu(p',p)=\int \mathrm{d}W\rho(W)\frac{1}{\gamma p'-W}\gamma_\mu\frac{1}{\gamma p-W} \tag{15}$$

as the longitudinal part, obeying identity (7), it is possible to prove that the remainder G^{T} in the second gauge approximation (13) is non-singular† as the photon momentum vanishes. More particularly:

$$\lim_{p\to p'} G^{\mathrm{T}}_{\mu}(p',p)=0.$$

(Appendix 1 gives an explicit demonstration.) Furthermore, the approximated vertex incorporates full second-order perturbation theory via the direct substitution $\rho(W)=\delta(W-m)$.

One may tidy up the appearance of (13) by using the dispersion representation

$$\Sigma(p|W)=-\frac{1}{\pi}\int\frac{\operatorname{Im}\Sigma(W'|W)\,\mathrm{d}W'}{\gamma p-W'} \tag{16}$$

whereupon

$$G_{\mu}(p',p)=\int \mathrm{d}W\frac{\rho(W)}{\gamma p'-W}\left(\gamma_{\mu}+\Lambda^{2}_{\mu}(p',p|W)+\int\frac{\mathrm{d}W'}{\pi}\frac{\operatorname{Im}\Sigma^{2}(W'|W)}{\gamma p'-W'}\gamma_{\mu}\frac{1}{\gamma p-W'}\right)$$
$$\times\frac{1}{\gamma p-W}+\mathrm{O}(e^{4}). \tag{13a}$$

The spectral function equation comes by inserting (13) into (4)

$$\int \mathrm{d}W\frac{\rho(W)(W-m_0)}{\gamma p-W}$$
$$=\mathrm{i}e^{2}\int \bar{\mathrm{d}}^{4}k\,D^{\kappa\lambda}(k)\int\frac{\mathrm{d}W\rho(W)}{\gamma p-W}$$
$$\times\left(\gamma_{\kappa}+\Lambda^{2}_{\kappa}(p,p-k|W)+\int\frac{\mathrm{d}W'}{\pi}\frac{\operatorname{Im}\Sigma^{2}(W'|W)}{\gamma p-W'}\gamma_{\kappa}\frac{1}{\gamma(p-k)-W'}\right)$$
$$\times\frac{1}{\gamma(p-k)-W}\gamma_{\lambda}$$
$$+G^{\mathrm{T}}_{\mu\nu}\text{ contributions of }\mathrm{O}(e^{6}) \tag{17}$$

and may be linearised by leaving the photon undressed.

$$D^{\kappa\lambda}(k)\to(-\eta^{\kappa\lambda}+(1-a)k^{\kappa}k^{\lambda}/k^{2})/k^{2},$$

which implies that the (logarithmic, ultraviolet) effects of vacuum polarisation on electron propagation are being neglected. That being so, ρ satisfies the standard equation (Delbourgo and West 1977a):

$$\int \mathrm{d}W\rho(W)[W-m_0+\Sigma(p|W)]/(\gamma p-W)=0$$
$$+\text{terms of order } e^{6} \tag{17a}$$

where

$$\Sigma(p|W)=\Sigma^{2}(p|W)+\Sigma^{4}(p|W) \tag{18}$$

† In this respect we disagree with King (1983) who includes just the last two terms of (13) as part of his longitudinal vertex and then must *impose* $G^{\mathrm{T}}(p,p;0)=0$ through some 'regularisation'. In our case the vanishing of the transverse vertex is automatic.

and

$$\Sigma^4(p|W) = -\mathrm{i}e^2 \int \bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)\left(\Lambda^2_\kappa(p, p-k|W) + \int \frac{\mathrm{d}W'}{\gamma p - W'}\gamma_\kappa \right.$$
$$\left. \times \frac{\operatorname{Im}\Sigma^2(W'|W)/\pi}{[\gamma(p-k) - W'][\gamma(p-k) - W]}\right)\gamma_\lambda \tag{18a}$$

is associated with a purely transverse vertex. Taking the discontinuity of (17a) it follows that

$$\varepsilon(W)\rho(W)[W - m_0 + \Sigma(W|W)] = \int \mathrm{d}W' \frac{\rho(W')}{\pi}\frac{\operatorname{Im}\Sigma(W|W')}{W - W'} + \mathrm{O}(e^6). \tag{19}$$

The evaluation of Σ^2 and Σ^4 is so involved that we have relegated it to appendix 2 and have only outlined the essential details at that. Here we shall require formulae (A2.1), (A2.2), (A2.7), (A2.8)–(A2.11). Since $m_0 = \operatorname{Re}\,\Sigma(m, m)$, this still leaves a wavefunction renormalisation infinity on the LHS of (19) of order e^4 which we anticipate should cancel against an infinity of the same order on the right since ρ is renormalised. The cancellation happens in any perturbative expansion for the ρ and it happens for us as well (because we incorporate perturbation theory exactly up to order e^4, barring internal vacuum polarisation). Thus†

$$\operatorname{Im}\Sigma^4(W|m) \supset [\operatorname{Im}\Sigma^2(W|m)/(W-m)][\Sigma^2(W|W) - \Sigma^2(m|m)] \tag{20}$$

ensures that the divergent log Λ^2 match up on each side of (19). Consequently, in the second gauge approximation, we finally have the finite renormalised equation:

$$\pi\varepsilon(W)\rho(W)(W-m)\left[1 + \frac{e^2}{16\pi^2}\left(1 + \frac{3W}{W-m}\ln\left(\frac{W^2}{m^2}\right)\right)\right]$$
$$= \int \mathrm{d}W'\,\rho(W')\frac{\operatorname{Im}\Sigma_{\mathrm{f}}(W|W')}{W - W'} \tag{21}$$

where the logarithmic divergences are absent in Σ_{f}. Specifically, using the abbreviations $x = W'/W$, $\eta = e^2/16\pi^2$, the kernel on the RHS of (21) reads

$$\operatorname{Im}\Sigma_{\mathrm{f}}(W|W')/\pi w\eta$$
$$= (1-x^2)\theta(1-x^2)[a(1+x^2) - x(a+3)]$$
$$\times\left\{1 - 4\eta\ln x^2 + \eta(1+x)\right.$$
$$\left.\times\left[1 + \frac{2(1-4x)}{1-x^2}\ln x^2 + 2(1-4x+x^2)\ln\left(\frac{1-x^2}{x^2}\right)\right]\right\}$$
$$+ 4\eta\theta(1-x^2)[x(1+x^2+2x^3)Z_1 - (1-x-x^2)Z_2]$$
$$+ \eta\theta(1-9x^2)[xY_1 + Y_2] - \eta(1-x^2)\theta(1-x^2)\left\{-\tfrac{7}{2} + 24x - \tfrac{1}{2}x^2 - 2x^3\right.$$

† $\Sigma^2(W|W) = -(3e^2W/16\pi^2)[\ln(\Lambda^2/W^2) + 1]$ independently of the gauge parameter a, and our calculations indeed confirm that

$$\operatorname{Im}\Sigma^4(p|W) \supset -(3e^4/256\pi^2)(p+W)\{ap[1 + (W^2/p^2)] - (a+3)W\}\{p[\ln(\Lambda^2/p^2) + 1] - W[\ln(\Lambda^2/W^2) + 1]\}$$

$$+a(-3+x+2x^2+x^3)+[15x^3+a(1-4x)(1+x^3)]\ln x^2/(1-x^2)$$
$$+2[(1-a)(-1+4x-x^2-x^3+4x^4-x^5)+(-6x+8x^2)$$
$$\times(1-x^2)]\ln\left(\frac{1-x^2}{x^2}\right)+\frac{4x(1+x)}{1-x}\left[f\left(\frac{1}{1-x^2}\right)-\ln x^2\ln\left(\frac{1-x^2}{x^2}\right)\right]\Bigg\}. \quad (22)$$

In (22) we recall that f is the dilogarithmic function (see (A2.10)) and that (Z_1, Z_2) and (Y_1, Y_2) correspond to electron–two photon and two electron–positron cut contributions, respectively, expressed in dimensionless variables, integral representations for which are stated in appendix 2.

Equation (21) is obviously extremely difficult to solve. However, some real simplifications occur in the infrared limit as we might expect. Let $\omega \equiv W/m$, then as $\omega \to 1$, the LHS of the equation tends to

$$\pi\varepsilon(\omega)\rho(\omega)(\omega-1)[1+7\eta].$$

In that limit $f \to x^2-1$, $Z_1 \to 1-x$, $Z_2 \to 2(1-x)^2$, Y_1, $Y_2 \to 0$, so the kernel (22) 'miraculously' tends to

$$2\eta(1-x)(a-3)(1+7\eta).$$

As a consequence, a remarkable cancellation of the $(1+7e^2/16\pi^2)$ factor on each side leaves us with

$$(\omega-1)\rho(\omega) \approx 2\eta(a-3)\int_1^\omega d\omega'\,\rho(\omega'),$$

and the standard infrared result:

$$\rho(W) \approx (W-m)^{-1+(a-3)e^2/8\pi^2}. \quad (23)$$

The fact that this behaviour emerges properly is a useful check that our kernel (22) is right.

We may carry out a similar analysis in the ultraviolet domain $\omega \gg 1$. Here we have to be more careful with the asymptotic estimation as typical integrals

$$\int \frac{\ln(W'/W)}{W'-W}\,dW'$$

must be treated with care. Because η is so small one can extract the dominant behaviour through the ansatz (Parker 1984)

$$\rho(W)=\varepsilon(W)[W(W^2)^{k_1}(1+b\ln W^2)+m(W^2)^{k_2}(1+c\ln W^2)] \quad (24)$$

and find self-consistency in QED with

$$k_1=-1+a\eta, \qquad k_2=-1-3\eta, \qquad b=c=-3\eta$$

leading to the expression for $S(p)$ quoted in the introduction. It would be misleading not to mention that the logarithmic term is undoubtedly influenced by internal vacuum polarisation corrections so the coefficient b will correspondingly alter. Moreover, higher orders of gauge approximation will bring in further powers of $(\eta \ln W^2)$; we are ignorant about the coefficients of these terms and whether the leading logs can be summed.

To find the complete analytical solution of (21) is probably impossible but it should be feasible to obtain a numerical solution for $\rho(W)$ and hence for $S(p)$ in any gauge. This task remains to be done and will be reported separately. Knowledge of ρ is vital for seeing how the gauge independence of vacuum polarisation is substantiated at least to order e^4, and thus how reliable are the results on dynamically generated vector boson masses (Delbourgo *et al* 1982). Also it will provide a non-perturbative approximation for the off-shell charge and magnetic form factors when one integrates over W in equation (13), which may be testable against known experimental data.

4. Scalar electrodynamics

We round off our study by treating the parallel case of scalar electrodynamics. Although Parker (1984) did examine the problem in the Fermi gauge ($a=1$) there remain questions associated with gauge dependence that can only be answered by finding the results for arbitrary a. We sketch the computations below.

All equations up to (11), excepting (4)–(6), can be taken over by the straight replacements $\rho(w)\to\rho(W^2)$, $S(p)\to\Delta(p)$, $(\gamma p-m)\to(p^2-m^2)$, etc. In place of (4), (5) and (12) we have

$$\begin{aligned}\Delta(p)(p^2-m_0^2)=Z^{-1}&+\mathrm{i}e^2\int \bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)G_\kappa(p,p-k;k)(2p-k)_\lambda\\ &-e^4\int \bar{\mathrm{d}}^4k\,\bar{\mathrm{d}}^4k'\;\eta_{\kappa\lambda}D^{\kappa\mu}(k')D^{\lambda\nu}(p-k-k')G_{\mu\nu}(p-k-k',p;k,k')\\ &+\text{tadpole term.}\end{aligned}\tag{4s}$$

$$\begin{aligned}&[(p'^2-p^2)+\text{tadpole}]G_\mu(p',p)\\ &\quad=(p'+p)_\mu\Delta(p)-2\mathrm{i}e^2\int \bar{\mathrm{d}}^4k\,D_{\mu\lambda}(k)G^\lambda(p-k,p;-k)\\ &\quad\quad-\mathrm{i}e^2\int \bar{\mathrm{d}}^4k(2p'-k)_\kappa D^{\kappa\lambda}(k)G_{\lambda\mu}(p'-k,p;-k,p'-p)\\ &\quad\quad+e^4\int \bar{\mathrm{d}}^4k\,\bar{\mathrm{d}}^4k'\;D^\lambda_\rho(k')D^{\rho\kappa}(k)G_{\lambda\kappa\mu}(p'-k-k',p;-k,-k',p'-p).\end{aligned}\tag{5s}$$

$$G^{\mathrm{L}}_{\nu\mu}=\int\frac{\mathrm{d}W^2\rho(W^2)}{(p'^2-p^2)(p^2-W^2)}\left(2\eta_{\mu\nu}-\frac{(2p'+k')_\nu(2p+k)_\mu}{(p+k)^2-W^2}-\frac{(2p'-k)_\mu(2p-k')_\nu}{(p-k')^2-W^2}\right).\tag{12s}$$

In the second gauge approximation, and exactly to order e^4 in (4s), we may use (12s) on the RHS of (5s) and drop $G_{\lambda\kappa\mu}$. This gives the analogue of (13), namely,

$$\begin{aligned}G_\mu(p',p)=\int\frac{\mathrm{d}W^2\,\rho(W^2)}{(p'^2-W^2)(p^2-W^2)}\bigg(&(p'+p)_\mu+\Lambda^2_\mu(p',p|W)\\ &+\frac{\Pi^2(p|W)-\Pi^2(p'|W)}{p'^2-p^2}(p'+p)_\mu\bigg)\end{aligned}\tag{13s}$$

where

$$\Lambda^2_\mu(p', p|W) = \mathrm{i}e^2 \int \bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)\left[\frac{(2p'+k)_\lambda(p+p'+2k)_\mu(2p+k)_\kappa}{[(p'+k)^2-W^2][(p+k)^2-W^2]}\right.$$
$$\left. -2\eta_{\mu\lambda}\left(\frac{(2p'+k)_\lambda}{(p'+k)^2-W^2}+\frac{(2p+k)_\lambda}{(p+k)^2-W^2}\right)\right] \tag{14s}$$

and

$$\Pi^2(p|W) = -\mathrm{i}e^2 \int \bar{\mathrm{d}}^4k\, D^{\kappa\lambda}(k)\,\frac{(2p+k)_\kappa(2p+k)_\lambda}{(p+k)^2-W^2} \tag{14s$'$}$$

are the order e^2 vertex and self-energy respectively for a charged scalar meson of mass W. The last two terms in (13s) correspond to the transverse vertex correction (which again vanishes as $p'\to p$). If one substitutes these last equations into (4s) and takes the discontinuity, there results the renormalised equation

$$\pi\rho(W^2)(W^2-m^2)\left(1+\eta\frac{3W^2}{W^2-m^2}\ln\frac{W^2}{m^2}\right)$$
$$=\int_{m^2}^{W^2}\mathrm{d}W'^2\,\rho(W'^2)\,\frac{\operatorname{Im}\Pi_\mathrm{f}(W^2|W'^2)}{W^2-W'^2}. \tag{21s}$$

In arriving at (21s) we have cancelled off the e^4 divergences† on each side—there is a perfect match—to leave us with the finite kernel (here $u = W'^2/W^2$).

$$\operatorname{Im}\Pi_\mathrm{f}(W^2|W'^2)/\pi\eta W^2\theta(W^2-W'^2)$$
$$=(a-3)(1-u^2)+\eta I+\theta(1-9u)\eta k-\eta(1-u)$$
$$\times\left[\frac{75+143u}{4}-\frac{5+6u}{2(1-u)}-2(1+14u+u^2)\ln\left(\frac{1-u}{u}\right)\right.$$
$$+3(1-a)\left(1-u+\frac{(1-2u)(1+u)}{1-u}\ln u\right)$$
$$\left.+\frac{(1+3u)^2}{1-u}\left(3f\left(\frac{1}{u}\right)-2\ln u\ln\left(\frac{1-u}{u}\right)\right)\right] \tag{22s}$$

where

$$I=\int_u^1 \mathrm{d}u'(1-u')\{(4/u')(u'-u)[1+a+\tfrac{5}{8}(1-a)^2]$$
$$-(1/2u'^2)(3u'+3-2a)(3u'+3u-2au)\}$$

$$K=\int_0^{(1-\sqrt{u})^2}\mathrm{d}v\left(L+\frac{2(2+2u-v)(3+u-2v)}{v+u-1}\ln\left|\frac{L+v+u+1}{L-v-u+1}\right|\right)$$

† The divergent term is

$$\frac{e^4}{256\pi^3}(3-a)\left(1+\frac{m^2}{W^2}\right)\left(3\ln\frac{\Lambda^2}{m^2}+7\right)$$

if the interested reader would like to check it.

with

$$L=\left[\left(\frac{2-2u-v}{2+2u-v}\right)(1+u^2+v^2-2u-2v-2uv)\right]^{1/2}.$$

We have omitted the calculational details: suffice it to say that there are two photon–one meson cuts and a three-meson cut (in K).

If we approach the infrared region ($W^2\to m^2$ or $u\to 1$). The kernel simplifies to

$$\operatorname{Im}\Pi_f(W^2|W'^2)/\pi(W^2-W'^2)\to 2\eta(a-3)(1+3\eta)$$

and the equation tends to

$$\rho(W^2)(W^2-m^2)\approx 2\eta(a-3)\int_{m^2}^{W^2}\mathrm{d}W'^2\,\rho(W'^2)$$

producing the standard infrared result

$$\rho(W^2)\sim(W^2-m^2)^{-1+(a-3)e^2/8\pi^2}. \tag{23s}$$

Turning to the ultraviolet domain ($W^2/m^2\to\infty$ or $u\to 0$), we achieve asymptotic self-consistency with

$$\rho(W^2)\sim(W^2)^{-1+k\eta}(1+b\ln W^2)\quad\text{if}\quad k=(a-3)\quad\text{and}\quad b=-3\eta.$$

Here, again, the logarithmic term is affected by photon dressing and we cannot even hazard a guess about what happens in higher orders of gauge approximation. (The same criticism, incidentally, applies to Parker's results in the ultraviolet limit.)

We are slightly more hopeful that (21s) will yield itself to a full solution than the corresponding spinor equation (21). At all events, armed with known asymptotic solutions, we ought to be able to do a full numerical analysis of the spectral equations and go from there to determine the induced effects on vacuum polarisation. That work lies ahead.

Appendix 1

We want to prove that

$$G^{\mathrm{T}}_\mu(p',p)=\int \mathrm{d}W\frac{\rho(W)}{\gamma p'-W}\left[\Lambda^2_\mu(p',p|W)+\frac{\gamma p'\gamma_\mu+\gamma_\mu\gamma p}{p'^2-p^2}\Sigma^2(p|W)\right.$$
$$\left.-\Sigma^2(p'|W)\frac{\gamma p'\gamma_\mu+\gamma_\mu\gamma p}{p'^2-p^2}\right]\frac{1}{\gamma p-W}$$

vanishes as $p'\to p$. Consider the expression in the square brackets and let $\delta p=p'-p$ be small. Then to order $(\delta p)^0$

$$[\;]\approx \mathrm{i}e^2\int \bar{\mathrm{d}}^4k\,D^{\kappa\lambda}(k)\left[\gamma_\kappa\frac{1}{\gamma p-\gamma k-W}\gamma_\mu\frac{1}{\gamma p-\gamma k-W}\gamma_\lambda\right.$$
$$+\left(1+\delta p^\rho\frac{\partial}{\partial p^\rho}\right)\left(\gamma_\kappa\frac{1}{\gamma p-\gamma k-W}\gamma_\lambda\right)\frac{2p_\mu+\gamma\delta p\gamma_\mu}{2p\cdot\delta p}$$
$$\left.-\frac{2p_\mu+\gamma\delta p\gamma_\mu}{2p\cdot\delta p}\left(\gamma_\kappa\frac{1}{\gamma p-\gamma k-W}\gamma_\lambda\right)\right]$$

$$=\left(\eta_{\mu\rho}-\frac{p_\mu\delta p_\rho}{p\cdot\delta p}\right)\frac{\partial\Sigma^2(p|W)}{\partial p_\rho}-\left[\Sigma^2(p|W),\frac{\gamma\delta p\gamma_\mu}{2p\cdot\delta p}\right].$$

Define $\Sigma(p|W)=AW+\gamma pB$ where A and B are invariant scalar functions of p^2 and W^2. The derivatives of A and B give zero since p_ρ terms vanish upon contraction: also A drops out of the commutator term. Hence

$$[\quad]\approx B\left(\eta_{\mu\rho}-\frac{p_\mu\delta p_\rho}{p\cdot\delta p}\right)\gamma_\rho-B\left[\gamma p,\frac{\gamma\delta p\gamma_\mu}{2p\cdot\delta p}\right]\equiv 0$$

by γ-matrix algebra. This then confirms the finite behaviour of G^{T} and the dominance of G^{L} in the infrared, explaining why the gauge technique is so successful in that regime.

Appendix 2

The calculation of Σ^2 in (14′) is so straightforward that we shall just quote the answer:

$$\operatorname{Im}\Sigma^2(p|W)=\frac{e^2(p^2-W^2)}{16\pi p^4}\theta(p^2-W^2)[a\gamma p(p^2+W^2)-(a+3)Wp^2] \tag{A2.1}$$

$$\Sigma^2(p|W)=\frac{e^2}{16\pi^2}\left\{[a\gamma p-(a+3)W]\ln\frac{\Lambda^2}{W^2}+\gamma p\left(1-a+a\frac{W^2}{p^2}\right)\right.$$
$$\left.+\left(-a\gamma p\left[1+\frac{W^2}{p^2}\right]+[3+a]W\right)\left(1-\frac{W^2}{p^2}\right)\ln\left(1-\frac{p^2}{W^2}\right)\right\}. \tag{A2.2}$$

In equation (19) we also require the discontinuity of Σ^4 where

$$\Sigma^4(p|W)=-\mathrm{i}e^2\int \bar{\mathrm{d}}^4k\,D^{\kappa\lambda}(k)\left[\Lambda^2_\kappa(p,p-k|W)\right.$$
$$\left.+\int\frac{\mathrm{d}W'}{\gamma p-W'}\frac{\operatorname{Im}\Sigma^2(W'|W)}{\pi(W-W')}\gamma_\kappa\left(\frac{1}{\gamma(p-k)-W}-\frac{1}{\gamma(p-k)-W'}\right)\right]\gamma_\lambda$$
$$=-\mathrm{i}e^2\int\bar{\mathrm{d}}^4k\,D^{\kappa\lambda}(k)\Lambda^2_\kappa(p,p-k|W)\gamma_\lambda+\int\frac{\mathrm{d}W'\operatorname{Im}\Sigma^2(W'|W)}{\pi(\gamma p-W')(W-W')}$$
$$\times[\Sigma^2(p|W)-\Sigma^2(p|W')]$$
$$\equiv I_\Lambda+I_\Sigma \tag{A2.3}$$

The hard work commences now.

Decompose I_Λ into gauge-independent and gauge-dependent parts†

$$I_\Lambda=e^4\int\bar{\mathrm{d}}^4k\,\bar{\mathrm{d}}^4l\,\gamma_\kappa(\gamma p-\gamma k-W)^{-1}\gamma_\mu[\gamma(p-k-l)-W]^{-1}\gamma_\lambda(\gamma p-\gamma l-W)^{-1}$$
$$\times[-\eta^{\kappa\lambda}+(1-a)k^\kappa k^\lambda/k^2]/k^2l^2$$
$$\equiv I^i_\Lambda+I^g_\Lambda \tag{A2.4}$$

where

$$I^g_\Lambda=\mathrm{i}e^2(1-a)\int\bar{\mathrm{d}}^4k\,\gamma k(\gamma p-\gamma k-W)^{-1}[\Sigma^2_{a=0}(p-k|W)-\Sigma^2_{a=0}(p|W)] \tag{A2.5}$$

† $(1-a)ll/l^2$ disappear because Σ^4 comes from a transverse G_μ.

and

$$I^i_\Lambda = e^4 \int \bar{d}^4k\, \bar{d}^4l\, \gamma_\lambda(\gamma p - \gamma k - W)^{-1}\gamma_\mu[\gamma(p-k-l) - W]^{-1}\gamma^\lambda \times(\gamma p - \gamma l - W)^{-1}\gamma^\mu / k^2 l^2. \tag{A2.6}$$

From (A2.5) it is not very difficult to arrive at

$$\begin{aligned} \operatorname{Im} I^g_\Lambda = \frac{e^2(a-1)}{16\pi}\Big\{ &[\operatorname{Re}\Sigma^2_{a=0}(p|W) - \operatorname{Re}\Sigma^2_{a=0}(W|W)](1+\gamma p W^3/p^4) \\ &+\left(1-\frac{W^2}{p^2}\right)\left[\frac{\gamma p(p^2-5W^2)}{2p^2} + W - \frac{3Wp^2}{p^2-W^2}\ln\frac{p^2}{W^2}\right] \\ &+\operatorname{Im}\Sigma^2_{a=0}(p|W)\left[\ln\frac{\Lambda^2}{W^2} - 1 - \frac{\gamma p W}{p^2} - \left(1+\frac{\gamma p W^3}{p^4}\right)\ln\left|\frac{p^2-W^2}{W^2}\right|\right]\Big\} \end{aligned} \tag{A2.7}$$

but it is much harder to extract the imaginary part of (A2.6). We first disentangle the γ-algebra by writing

$$\begin{aligned} I^i_\Lambda = e^4 \int \bar{d}^4k\, \bar{d}^4l \{ &4W[(2p-k-l)(p-k-l) + (p-k)(p-l) - 2m^2] \\ &+ (4/\gamma p)[W^2 p(3p-2k-2l) - 2p(p-k-l)(p-k)(p-l)]\} \\ &\times\{[(p-k)^2 - W^2][(p-k-l)^2 - W^2][(p-l)^2 - W^2]k^2l^2\}^{-1} \end{aligned}$$

and then apply the Cutkosky–Nakanishi cutting rules: the imaginary parts then appear by letting various combinations of propagators go on-shell: either two photons and one electron, one photon and one electron, or two electrons and one positron. After much arduous computation one ends up with

$$\begin{aligned} &\operatorname{Im} I^i_\Lambda = W I^{(1)}_\Lambda + \gamma p I^{(2)}_\Lambda \\ &I^{(1)}_\Lambda = X_1 + (1 + W^2/p^2)Z_1 + Z_2 + Y_1 \\ &I^{(2)}_\Lambda = X_2 + (2W^4/p^4)Z_1 - (1 - W^2/p^2)Z_2 + Y_2 \end{aligned} \tag{A2.8}$$

where

$$\begin{aligned} X_1 &= -\frac{e^4}{32\pi^3\xi}\left\{\ln\frac{\Lambda^2}{W^2} + 1 + \frac{3}{2\xi}\ln(\xi-1) + (\xi - \tfrac{1}{2})\left[f(\xi) - \ln(\xi-1)\ln\left(\frac{\xi}{\xi-1}\right)\right]\right\} \\ X_2 &= -\frac{e^4}{32\pi^3\xi}\left\{-\frac{1}{4\xi}\ln\frac{\Lambda^2}{W^2} - \frac{1}{2} + \frac{1-\xi}{\xi^2}\ln(\xi-1) + \frac{(\xi-1)^2}{\xi}\right. \\ &\qquad\left.\times\left[f(\xi) - \ln(\xi-1)\ln\left(\frac{\xi}{\xi-1}\right)\right]\right\} \\ Z_1 &= \frac{e^4}{64\pi^3}\int_0^{(p-W)^2} dq^2 \frac{\ln\Phi}{p^2 - W^2 - q^2} \\ Z_2 &= \frac{e^4}{64\pi^3 p^2}\int_0^{(p-W)^2} dq^2 \ln\Phi \end{aligned} \tag{A2.9}$$

$$Y_1=\frac{e^4}{64\pi^3p^2}\int_{4W^2}^{(p-W)^2}\mathrm{d}q^2\,\frac{2q^2-p^2-5W^2}{p^2+3W^2-q^2}\ln\Psi$$

$$Y_2=\frac{e^4}{64\pi^3p^4}\int_{4W^2}^{(p-W)^2}\mathrm{d}q^2\,\frac{p^2q^2+5W^2q^2-q^4-4W^4-2p^2W^2}{p^2+3W^2-q^2}\ln\Psi$$

in which

$$\xi\equiv p^2/(p^2-W^2)$$

$$f(\xi)\equiv\int_0^1\frac{\mathrm{d}\xi'}{\xi-\xi'}\ln|\xi'-1|\qquad\text{(Spence's dilogarithm function),}$$

$$\Phi\equiv\left|\frac{p^2-q^2-W^2+\Delta}{p^2-q^2-W^2-\Delta}\right|;\qquad\Delta\equiv[p^4+q^4+W^4-2p^2W^2-2q^2W^2-2p^2q^2]^{1/2}$$

$$\Psi\equiv\left|\frac{p^2+3W^2-q^2+\Delta(1-4W^2/q^2)^{1/2}}{p^2+3W^2-q^2-\Delta(1-4W^2/q^2)^{1/2}}\right|.\qquad\text{(A2.10)}$$

The cuts in X and Z begin at $p^2=W^2$, those in Y at $p^2=9W^2$.

Finally we need the imaginary part of I_Σ. This is extracted from (A2.3) either through the discontinuity in the denominator multiplying $\mathrm{Im}\,\Sigma^2\times\mathrm{Re}\,\Sigma^2$ or as a principal value integral over $\mathrm{Im}\,\Sigma^2\times\mathrm{Im}\,\Sigma^2$. One eventually arrives at

$$\begin{aligned}\mathrm{Im}\,I_\Sigma=\frac{e^2}{16\pi^2}\,&\mathrm{Im}\,\Sigma^2(p|W)\left\{-4\ln\frac{\Lambda^2}{W^2}+2\left(1+\frac{W}{\gamma p}\right)\right.\\&\times\left[4W\left(\ln\frac{W^2}{p^2-W^2}+\frac{p^2}{p^2-W^2}\ln\frac{p^2}{W^2}\right)\right.\\&\left.\left.-\gamma p\left(\frac{p^2+W^2}{p^2}\ln\frac{W^2}{p^2-W^2}+\frac{p^2}{p^2-W^2}\ln\frac{p^2}{W^2}\right)\right]\right\}\\&+\frac{e^4}{256\pi^3}\,\frac{p^2-W^2}{p^2}\left(4W-\gamma p\frac{(p^2+W^2)}{p^2}\right)\left(a\ln\frac{\Lambda^2}{W^2}+1\right)\\&+\frac{e^4}{256\pi^3}\,\frac{a(p^2-W^2)}{p^2}\left[W\left(\frac{2W^2}{p^2}-\frac{4p^2}{p^2-W^2}\ln\frac{p^2}{W^2}\right)\right.\\&\left.+\gamma p\left(\frac{9p^2-7W^2}{2p^2}+\frac{p^4-4W^4}{p^2(p^2-W^2)}\ln\frac{p^2}{W^2}\right)\right]\\&-\frac{e^4}{64\pi^3}(a+3)W\frac{(p^2-W^2)}{p^2}\left[1-\frac{W^2}{p^2-W^2}\ln\frac{p^2}{W^2}\right].\end{aligned}\qquad\text{(A2.11)}$$

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